TRANSPORT PHENOMENA

MOMENTUM TRANSPORT

Velocity Distributions in Turbulent Flow
Introduction to Turbulent Flow

1. Comparisons of laminar and turbulent flows
2. Time-smoothed equations of change for incompressible fluids
3. The time-smoothed velocity profile near a wall
4. Empirical expressions for the turbulent momentum flux
5. Turbulent flow in ducts
Comparisons of Laminar and Turbulent Flows

We consider the flow in conduit of circular cross section. In a circular tube of radius $R$, for steady-state conditions and **laminar flow** we know that the velocity distribution and the average velocity are given by:

$$\frac{v_z}{v_{z,\text{max}}} = 1 - \left(\frac{r}{R}\right)^2 \quad \text{and} \quad \frac{\langle v_z \rangle}{v_{z,\text{max}}} = \frac{1}{2} \quad (\text{Re} < 2100)$$

and the pressure drop and mass rate $w$ are linearly related:

$$P_0 - P_L = \left(\frac{8\mu L}{\pi \rho R^4}\right)w \quad (\text{Re} < 2100)$$
Comparisons of Laminar and Turbulent Flows

For turbulent flow, the velocity is fluctuating with time chaotically at each point in the tube. We can measure a time-smoothed velocity at each point. The time-smoothed velocity and its average value are given by:

\[
\frac{\bar{v}_z}{v_{z,\text{max}}} \approx \left(1 - \frac{r}{R}\right)^{1/7} \quad \text{and} \quad \frac{\langle \bar{v}_z \rangle}{v_{z,\text{max}}} \approx \frac{4}{5} \quad (10^4 < \text{Re} < 10^5)
\]

Over the same range of Reynolds numbers the mass rate of flow and the pressure drop are no longer proportional but are related approximately by:

\[
P_0 - P_L \approx 0.198 \left(\frac{2}{\pi}\right)^{7/4} \left(\frac{\mu^{1/4}\lambda}{\rho R^{19/4}}\right) \omega^{7/4} \quad (10^4 < \text{Re} < 10^5)
\]
Comparisons of Laminar and Turbulent Flows

The laminar and turbulent velocity profiles are compared in graph below:
Comparisons of Laminar and Turbulent Flows

For the laminar flow around a flat plate, wetted on both sides, the solution of the boundary layer equations gave the drag force expression:

\[ F = 1.328 \sqrt{\rho \mu L W^2 v_\infty^3} \quad \text{(laminar)} \quad 0 < Re_L < 5 \times 10^5 \]

In which \( Re_L = L v_\infty \rho / \mu \) is the Reynolds number for a plate of length \( L \); the plate width is \( W \), and the approach velocity of the fluids is \( v_\infty \).

For turbulent flow, the dependence on the geometrical and physical properties is quite different:

\[ F \approx 0.74 \sqrt[5]{\rho^4 \mu L^4 W^5 v_\infty^9} \quad \text{(turbulent)} \quad (5 \times 10^5 < Re_L < 10^7) \]
Time-smoothed Equations of Change for Incompressible Fluids

We consider a turbulent flow in a tube with a constant pressure gradient. The velocity as a function of time is fluctuating in a chaotic fashion.

The fluctuations are irregular deviations from a mean value. The actual velocity can be regarded as the sum of the mean value and the fluctuation. For example, for the $z$-component of the velocity we write:

$$v_z = \bar{v}_z + v'_z$$

($\bar{v}_z$ does not depend on time)
Time-smoothed Equations of Change for Incompressible Fluids

The mean value is obtained from \( v_z(t) \) by making a time average over a large number of fluctuations is given by:

\[
\bar{v}_z = \frac{1}{t_0} \int_{t-\frac{1}{2}t_0}^{t+\frac{1}{2}t_0} v_z(s) \, ds
\]

The period \( t_0 \) being long enough to give a smooth averaged function.

The same comments we have made for velocity can also be made for pressure.

\( \bar{v}_z \) does depend on time. 
Time-smoothed Equations of Change for Incompressible Fluids

According to the previous equation we can find the following relations:

\[ \overline{v'_z} = 0 \quad \overline{v_z} = \overline{v_z} \quad \overline{v_zv'_z} = 0 \quad \frac{\partial}{\partial x} \overline{v_z} = \frac{\partial}{\partial x} \overline{v_z} \quad \frac{\partial}{\partial t} \overline{v_z} = \frac{\partial}{\partial t} \overline{v_z} \]

Having defined the time-smoothed quantities we can now move on to the time-smoothing equations of change. The equations of continuity and motion with \( \nu \) replaced by its equivalent \( \overline{\nu} + \nu' \).

The equation of continuity is then: \( (\nabla \cdot \nu) = 0 \)

\[ \frac{\partial}{\partial x} (\overline{v_x} + \nu'_x) + \frac{\partial}{\partial y} (\overline{v_y} + \nu'_y) + \frac{\partial}{\partial z} (\overline{v_z} + \nu'_z) = 0 \]
The $x$-component of the equation of motion in the $\partial/\partial t$ form:

$$
\frac{\partial}{\partial t} \rho(\vec{v}_x + v'_x) = -\frac{\partial}{\partial x} (\bar{p} + p') - \left( \frac{\partial}{\partial x} \rho(\vec{v}_x + v'_x)(\vec{v}_x + v'_x) + \frac{\partial}{\partial y} \rho(\vec{v}_y + v'_y)(\vec{v}_x + v'_x) \right)
$$

$$
+ \frac{\partial}{\partial z} \rho(\vec{v}_z + v'_z)(\vec{v}_x + v'_x) + \mu \nabla^2(\vec{v}_x + v'_x) + \rho g_x
$$

The $y$- and $z$-components of the equation of motion can be similarly written. We next time-smooth these equations which gives:

$$
\frac{\partial}{\partial x} \bar{v}_x + \frac{\partial}{\partial y} \bar{v}_y + \frac{\partial}{\partial z} \bar{v}_z = 0
$$

$$
\frac{\partial}{\partial t} \rho \bar{v}_x = -\frac{\partial}{\partial x} \bar{p} - \left( \frac{\partial}{\partial x} \rho \bar{v}_x \bar{v}_x + \frac{\partial}{\partial y} \rho \bar{v}_x \bar{v}_y + \frac{\partial}{\partial z} \rho \bar{v}_x \bar{v}_z \right)
$$

$$
- \left( \frac{\partial}{\partial x} \rho \bar{v'_x \bar{v}'_x} + \frac{\partial}{\partial y} \rho \bar{v'_y \bar{v}'_x} + \frac{\partial}{\partial z} \rho \bar{v'_z \bar{v}'_x} \right) + \mu \nabla^2 \bar{v}_x + \rho g_x
$$

These are the time-smoothed equations of continuity and motion for fluid with constant density and viscosity.
The Time-smoothed Velocity Profile near a Wall

We consider a time-smoothed velocity distribution in the neighborhood of a wall. We discuss several results: a Taylor expansion of the velocity near the wall, and the universal logarithmic and power law velocity distributions a little further out from the wall.

There are four different regions for describing turbulent flow near a wall: (1) viscous sublayer, (2) buffer layer, (3) inertial sublayer, (4) main turbulent stream.
TEMPERATURE DISTRIBUTION IN TURBULENT FLOW

13.1 Time-smoothed equations of change for incompressible nonisothermal flow

13.2 The time-smoothed temperature profile near a wall

13.3 Empirical expressions for the turbulent heat flux

13.4 Temperature distribution for turbulent flow in tubes

13.5 Temperature distribution for turbulent flow in jets
13.1 TIME- SMOOTHED EQUATIONS OF CHANGE FOR INCOMPRESSIBLE NONISOTHERMAL FLOW

\[ T = \bar{T} + T' \]

Where:

\( \bar{T} \) = time- smoothed temperature

\( T' \) = temperature fluctuation

Clearly \( T' \) averages to zero so that \( T' = 0 \), but quantities like \( \bar{v}_x T', \bar{v}_y T' \) and \( \bar{v}_z T' \) will not be zero because of the „correlation” between the velocity and temperature fluctuations at any point.
13.1 TIME-SMOOTHEOED EQUATIONS OF CHANGE FOR INCOMPRESSIBLE NONISOTHERMAL FLOW

For a nonisothermal pure fluid we need three equations of change. The time-smoothed equations of continuity and motion for a fluid with constant density and viscosity. For a fluid with constant $\hat{C}_p$, $\rho$, $\mu$ and $k$, when put in the d/dt form by using equation below:

$$\frac{\partial}{\partial t} (\rho f) + (\nabla \cdot \rho f) = \rho \frac{Df}{Dt}$$

, and with Newton’s and Fourier’s Law included, becomes:

$$\frac{\partial}{\partial t} \rho \hat{C}_p T = - \left( \frac{\partial}{\partial x} \rho \hat{C}_p v_x T + \frac{\partial}{\partial y} \rho \hat{C}_p v_y T + \frac{\partial}{\partial z} \rho \hat{C}_p v_z T \right) + k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

$$+ \mu \left[ 2 \left( \frac{\partial v_x}{\partial x} \right)^2 + \left( \frac{\partial v_y}{\partial y} \right)^2 + 2 \left( \frac{\partial v_x}{\partial y} \right) \left( \frac{\partial v_y}{\partial x} \right) + \cdots \right]$$

In which only few sample terms in the viscous dissipation term have been written. $- (\tau : \nabla \mathbf{v}) = \mu \Phi_v$
13.1 TIME-SMOOTHED EQUATIONS OF CHANGE FOR INCOMPRESSIBLE NONISOHERMAL FLOW

In equation:

\[
\frac{\partial}{\partial t} \rho \hat{C}_p T = -\left( \frac{\partial}{\partial x} \rho \hat{C}_p v_x T + \frac{\partial}{\partial y} \rho \hat{C}_p v_y T + \frac{\partial}{\partial z} \rho \hat{C}_p v_z T \right) + k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) \\
+ \mu \left[ 2 \left( \frac{\partial v_x}{\partial x} \right)^2 + \left( \frac{\partial v_y}{\partial y} \right)^2 + 2 \left( \frac{\partial v_x}{\partial y} \right) \left( \frac{\partial v_y}{\partial x} \right) + \cdots \right]
\]

we replace \( T \) by \( T = \overline{T} + T' \), \( v_x \) by \( \overline{v}_x + v'_x \) and so on. The equation is time-smoothed to give:

\[
\frac{\partial}{\partial t} \rho \hat{C}_p \overline{T} = -\left( \frac{\partial}{\partial x} \rho \hat{C}_p \overline{v}_x \overline{T} + \frac{\partial}{\partial y} \rho \hat{C}_p \overline{v}_y \overline{T} + \frac{\partial}{\partial z} \rho \hat{C}_p \overline{v}_z \overline{T} \right) \\
- \left( \frac{\partial}{\partial x} \rho \hat{C}_p v'_x T' + \frac{\partial}{\partial y} \rho \hat{C}_p v'_y T' + \frac{\partial}{\partial z} \rho \hat{C}_p v'_z T' \right) \\
+ k \left( \frac{\partial^2 \overline{T}}{\partial x^2} + \frac{\partial^2 \overline{T}}{\partial y^2} + \frac{\partial^2 \overline{T}}{\partial z^2} \right) \\
+ \mu \left[ 2 \left( \frac{\partial \overline{v}_x}{\partial x} \right)^2 + \left( \frac{\partial \overline{v}_y}{\partial y} \right)^2 + 2 \left( \frac{\partial \overline{v}_x}{\partial y} \right) \left( \frac{\partial \overline{v}_y}{\partial x} \right) + \cdots \right] \\
+ \mu \left[ 2 \left( \frac{\partial v'_x}{\partial x} \right) \left( \frac{\partial v'_x}{\partial x} \right) + \left( \frac{\partial v'_y}{\partial y} \right) \left( \frac{\partial v'_y}{\partial y} \right) + 2 \left( \frac{\partial v'_x}{\partial y} \right) \left( \frac{\partial v'_y}{\partial x} \right) + \cdots \right]
\]
Comparison of this equation with the preceding one shows that the time-smoothed equation has the same form as the original equation, except for the appearance of the terms indicated by dashed underlines, which are concerned with the turbulent fluctuations. We are thus led to the definition of the turbulent heat flux with components:

\[
\overline{q^{(t)}} = \rho \hat{C}_p \overline{u'v'T'}
\]

And the turbulent energy dissipation function

\[
\overline{\Phi^{(t)}} = \sum_{i=1}^{3} \sum_{j=1}^{3} \left( \frac{\partial v'_i}{\partial x_j} \frac{\partial v'_j}{\partial x_i} + \frac{\partial v'_j}{\partial x_i} \frac{\partial v'_i}{\partial x_j} \right)
\]
13.1 TIME-SMOOTHED EQUATIONS OF CHANGE FOR INCOMPRESSIBLE NONISOTHERMAL FLOW

To summarize, we list all three-smoothed equations of change for turbulent flows of pure fluids with constant $\mu$, $\rho$, $\hat{C}_p$ and $k$ in their $D/Dt$ form:

\[ (\nabla \cdot \bar{v}) = 0 \quad (13.1-6) \]
\[ \rho \frac{D\bar{v}}{Dt} = -\nabla \bar{p} - [\nabla \cdot (\tau^{(v)} + \bar{\tau}^{(v)})] + \rho \bar{g} \quad (13.1-7) \]
\[ \rho \hat{C}_p \frac{DT}{Dt} = - (\nabla \cdot (\bar{q}^{(v)} + \bar{q}^{(t)})) + \mu (\bar{\Phi}^{(v)} + \Phi^{(t)}) \quad (13.1-8) \]

Which is understood that:

\[ D/Dt = \partial/\partial t + \bar{v} \cdot \nabla. \]

and where:

\[ \bar{q}^{(v)} = -k \nabla \bar{T} \]
13.2 THE TIME-SMOOTHED TEMPERATURE PROFILE NEAR A WALL

We consider the turbulent flow along a flat wall as shown in figure below, and we inquire as to the temperature in the inertial sublayer. We pattern the development after that for:

\[
\frac{d \overline{v_y}}{dy} = \frac{1}{\kappa} \sqrt{\frac{\tau_0}{\rho y}}
\]

We let the heat flux into the fluid at \( y=0 \) be

\[
q_0 = \overline{q_y} \big|_{y=0}
\]

and we postulate that the heat flux in the inertial sublayer will not be very different from that at the wall.

Fig. 13.2-1. Temperature profile in a tube with turbulent flow. The regions are (1) viscous sublayer, (2) buffer layer, (3) inertial sublayer, and (4) main turbulent stream.
13.2 THE TIME-SMOOTHED TEMPERATURE PROFILE NEAR A WALL

We seek to relate $q_0$ to the time-smoothed temperature gradient in the inertial sublayer.

We must further use the fact that the linearity of the energy equation implies that $dT/dy$ must be proportional to $q_0$.

The only combination that satisfies these requirements is:

$$\frac{-dT}{dy} = \frac{\beta q_0}{\kappa \rho \hat{C}_p \nu_* y}$$

Where $k$ is the dimensionless constant, and $\beta$ is an additional constant (which turns out to be the turbulent Prandtl number)

$$Pr^{(t)} = \frac{\nu^{(t)}}{\alpha^{(t)}}$$

When we integrate last equation we get:

$$T_0 - \bar{T} = \frac{\beta q_0}{\kappa \rho \hat{C}_p \nu_*} \ln y + C$$

Where: $T_0$ is the wall temperature and $C$ is a constant of integration.
13.2 THE TIME-SMOOTHED TEMPERATURE PROFILE NEAR A WALL

In addition, if we introduce the dimensionless coordinate $y\nu_* / \nu$, then last equation can be rewritten as:

$$T_0 - \bar{T} = \frac{\beta q_0}{\kappa \rho C_p \nu_*} \left[ \ln \left( \frac{y \nu_*}{\nu} \right) + f(Pr) \right]$$

for $\frac{y \nu_*}{\nu} > 1$

in which $f(Pr)$ is a function representing the thermal resistance between the wall and the inertial sublayer.

Figure 13.2-1. Temperature profile in a tube with turbulent flow. The regions are (1) viscous sublayer, (2) buffer layer, (3) inertial sublayer, and (4) main turbulent stream.
13.3 EMPIRICAL EXPRESSIONS FOR THE TURBULENT HEAT FLUX

The time-smoothing of the energy equation gives rise to a turbulent heat flux. In order to solve the energy equation for the time-smoothed temperature profiles, it is customary to postulate a relation between a turbulent heat flux and the time-smoothed temperature gradient. We summarize two of the most popular empirical expressions.

Eddy Thermal Conductivity

By analogy with the Fourier law of heat conduction we may write:

\[ \overline{q}_y^{(t)} = -k^{(t)} \frac{dT}{dy} \]

In which the \( k^{(t)} \) the turbulent thermal conductivity or the eddy thermal conductivity. This quantity depends on position, direction and the nature of the turbulent flow.
13.3 EMPIRICAL EXPRESSIONS FOR THE TURBULENT HEAT FLUX

The eddy kinematic viscosity $\nu^{(t)} = \mu^{(t)}/\rho$ and the eddy thermal diffusivity have the same dimensions. $\alpha^{(t)} = k^{(t)}/\rho c_p$

Their ratio is a dimensionless group called the *turbulent Prandtl number*.

$$Pr^{(t)} = \frac{\nu^{(t)}}{\alpha^{(t)}}$$

**The Mixing- Length Expression of Prandtl and Taylor**

According to Prandtl’s mixing- length theory, monumentum and energy are transferred in turbulent flow by the same mechanism. We obtain:

$$\bar{q}_y^{(t)} = -\rho \hat{c}_p l^2 \left| \frac{d\bar{v}_x}{dy} \right| \frac{dT}{dy}$$

Where $l$ is the Prandtl length.
13.4 TEMPERATURE DISTRIBUTION FOR TURBULENT FLOW IN TUBES

The fluid enters the tube of radius R at an inlet temperature $T_1$. For $z > 0$ the fluid is heated because of a uniform radial heat flux $q_0$ at the wall (figure below).

We start from the energy equation, written in cylindrical coordinates:

$$\rho \hat{C}_p v_z \frac{\partial \bar{T}}{\partial z} = -\frac{1}{r} \frac{\partial}{\partial r} \left( r(\bar{q}^{(r)}_r + \bar{q}^{(r)}) \right)$$

Then insertion of expression:

$$\bar{q}_r = -(k + k^{(r)}) \frac{dT}{dr} = -(1 + \frac{\alpha^{(r)}}{\alpha})k \frac{dT}{dr} = +(1 + \frac{\nu^{(r)}}{\alpha})k \frac{dT}{dy}$$

gives:

$$v_z \frac{\partial \bar{T}}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r} \left( r(\alpha + \alpha^{(r)}) \frac{\partial \bar{T}}{\partial r} \right)$$

Figure 13.4-1. System used for heating a liquid in fully developed turbulent flow with constant heat flux for $z > 0$. 
13.4 TEMPERATURE DISTRIBUTION FOR TURBULENT FLOW IN TUBES

This is to be solved with the boundary conditions:

\[
\begin{align*}
\text{at } r = 0, & \quad \overline{T} = \text{finite} \\
\text{at } r = R, & \quad +k \frac{\partial \overline{T}}{\partial r} = q_0 \\
\text{at } z = 0, & \quad \overline{T} = T_1 \\
\end{align*}
\]

Then the last equation in dimensionless form is:

\[
\phi \frac{\partial \Theta}{\partial \xi} = \frac{1}{\xi} \frac{\partial}{\partial \xi} \left( \xi \left(1 + \frac{\alpha^{(0)}}{\alpha} \right) \frac{\partial \Theta}{\partial \xi} \right)
\]

In which \( \phi(\xi) = \overline{v}_z / \overline{v}_{\text{max}} \) e dimensionless turbulent velocity profile. This equation is to be solved with the dimensionless boundary conditions:

\[
\begin{align*}
\text{at } \xi = 0, & \quad \Theta = \text{finite} \\
\text{at } \xi = 1, & \quad \frac{\partial \Theta}{\partial \xi} = 1 \\
\text{at } \zeta = 0, & \quad \Theta = 0 \\
\end{align*}
\]
13.4 TEMPERATURE DISTRIBUTION FOR TURBULENT FLOW IN TUBES

The resulting equation for $\psi$ is:

$$\frac{1}{\xi} \frac{d}{d\xi} \left( \xi \left(1 + \frac{\alpha^{(t)}}{\alpha}\right) \frac{d\psi}{d\xi} \right) = C_0 \phi$$

Integrating this equation twice and then constructing the function $\Theta$ using equation:

$$\Theta(\xi, \zeta) = C_0 \zeta + \Psi(\xi)$$

We get:

$$\Theta = C_0 \zeta + C_0 \int_0^\xi \frac{I(\tilde{\xi})}{\xi[1 + (\alpha^{(t)}/\alpha)]} d\tilde{\xi} + C_1 \int_0^\xi \frac{1}{\xi[1 + (\alpha^{(t)}/\alpha)]} d\tilde{\xi} + C_2$$

is $\Theta(\xi)$ shorthand for the integral.

$$I(\bar{\xi}) = \int_0^{\bar{\xi}} \phi \bar{\xi} d\bar{\xi}$$
13.4 TEMPERATURE DISTRIBUTION FOR TURBULENT FLOW IN TUBES

The constant of integration $C_1$ is set to zero in order to satisfy boundary condition 1. The constant $C_0$ is found by applying boundary condition 2, which gives:

$$C_0 = \left( \int_0^1 \phi \xi \, d\xi \right)^{-1} = [I(1)]^{-1}$$

We next get an expression for the dimensionless temperature difference $\Theta_0 - \Theta_b$, the „driving force” for the heat transfer at the tube wall:

$$\Theta_0 - \Theta_b = C_0 \int_0^1 \frac{I(\xi)}{\xi[1 + (\alpha^{(t)}/\alpha)]} \, d\xi - \frac{C_0}{I(1)} \int_0^1 \phi \xi \left[ \int_0^\xi \frac{I(\bar{\xi})}{\bar{\xi}[1 + (\alpha^{(t)}/\alpha)]} \, d\bar{\xi} \right] d\xi$$

$$= C_0 \int_0^1 \frac{I(\xi)}{\xi[1 + (\alpha^{(t)}/\alpha)]} \, d\xi - \frac{C_0}{I(1)} \int_0^1 \frac{I(\bar{\xi})}{\bar{\xi}[1 + (\alpha^{(t)}/\alpha)]} \left[ \int_{\bar{\xi}}^1 \phi \xi \, d\xi \right] d\bar{\xi}$$
13.4 TEMPERATURE DISTRIBUTION FOR TURBULENT FLOW IN TUBES

When this equation is used:

\[ C_0 = \left( \int_0^1 \phi \xi d\xi \right)^{-1} = [I(1)]^{-1} \]

we get:

\[ \Theta_0 - \Theta_b = \int_0^1 \frac{[I(\xi)/I(1)]^2}{\xi[1 + (\alpha^{(n)}/\alpha)]} d\xi \]

But the quantity \( I(1) \) appearing in equation above has a simple interpretation:

\[ I(1) = \int_0^1 \phi \xi d\xi = \left( \int_0^R \overline{v}_z r dr \right) \frac{1}{\overline{v}_{z,\text{max}} R^2} = \frac{1}{2} \frac{\langle \overline{v}_z \rangle}{\overline{v}_{z,\text{max}}} \]

Finally, we want to get the dimensionless wall heat flux,

\[ \frac{q_0 D}{k(T_0 - T_p)} = \frac{2}{\Theta_0 - \Theta_b} \]

The reciprocal of which is:

\[ \frac{k(T_0 - T_p)}{q_0 D} = 2 \left( \frac{\overline{v}_{z,\text{max}}}{\langle \overline{v}_z \rangle} \right)^2 \int_0^1 \frac{[I(\xi)]^2}{\xi[1 + (\nu^{(n)}/\nu)(Pr/Pr^{(n)})]} d\xi \]
13.4 TEMPERATURE DISTRIBUTION FOR TURBULENT FLOW IN TUBES

To use this result, it is necessary to have an expression for the time-smoothed velocity distribution, the turbulent kinematic viscosity as a function of position, and a postulate for the turbulent Prandtl number Pr.
Concentration Distributions in Turbulent Flow
Concentration fluctuations and the time-smoothed concentration

\[ C_A = C_A + C'_A \]

**Time-smoothed for tube**

In tube flow with mass transfer at the wall, one expects that the time-smoothed concentration \( C_a \) will vary only slightly with position in the turbulent core, where the transport by turbulent eddies predominates.

In the slowly moving region near the boundary surface, on the other hand, the concentration \( C_a \) will be expected to change within a small distance from its turbulent-core value to the wall value.
Time-smoothing of the equation of continuity of A

Using equation:

$$\rho \left( \frac{\partial \omega_A}{\partial t} + (\mathbf{v} \cdot \nabla \omega_A) \right) = \rho \mathcal{D}_{AB} \nabla^2 \omega_A + r_A$$

We get the equation:

$$\frac{\partial c_A}{\partial t} = - \left( \frac{\partial}{\partial x} v_x c_A + \frac{\partial}{\partial y} v_y c_A + \frac{\partial}{\partial z} v_z c_A \right) + \mathcal{D}_{AB} \left( \frac{\partial^2 c_A}{\partial x^2} + \frac{\partial^2 c_A}{\partial y^2} + \frac{\partial^2 c_A}{\partial z^2} \right) - k_{ii} c_A^n$$

$k$ - reaction rate coefficient for the chemical reaction;

When $Ca$ is replaced by $CA + CA$ and $Vi$ by $Vi + Vi$, we obtain after time-averaging.

$$\frac{\partial c_A}{\partial t} = - \left( \frac{\partial}{\partial x} \overline{v_x c_A} + \frac{\partial}{\partial y} \overline{v_y c_A} + \frac{\partial}{\partial z} \overline{v_z c_A} \right) - \left( \frac{\partial}{\partial x} \overline{v_x c_A^r} + \frac{\partial}{\partial y} \overline{v_y c_A^r} + \frac{\partial}{\partial z} \overline{v_z c_A^r} \right)$$

$$+ \mathcal{D}_{AB} \left( \frac{\partial^2 \overline{c_A}}{\partial x^2} + \frac{\partial^2 \overline{c_A}}{\partial y^2} + \frac{\partial^2 \overline{c_A}}{\partial z^2} \right) - \begin{cases} k_{1} \overline{c_A^n} & \text{or} \\ k_{2} (\overline{c_A^2} + \overline{c_A^3}) & \end{cases}$$
The third of the turbulent fluxes:

- Turbulent molar flux (vector): \( \bar{J}_{Ai}^{(t)} = \bar{v}_i \bar{c}_A^t \)
- Turbulent momentum flux (tensor): \( \bar{\tau}_{ij}^{(t)} = \rho \bar{v}_i \bar{v}_j^t \)
- Turbulent heat flux (vector): \( \bar{q}_i^{(t)} = \rho C_p \bar{v}_i T^t \)

When we summarise all three of the time-smoothed equations of change for turbulent flow of an isothermal, binary fluid mixture with constant \( p, D_{AB}, \) and \( \mu \):

**continuity:**
\[ (\nabla \cdot \bar{v}) = 0 \]

**motion:**
\[ \rho \frac{D\bar{v}}{Dt} = -\nabla \bar{p} - \left[ \nabla \cdot (\bar{\tau}^{(v)} + \bar{\tau}^{(t)}) \right] + \rho g \]

**continuity of A:**
\[ \frac{D\bar{c}_A}{Dt} = -\left( \nabla \cdot (\bar{J}_A^{(v)} + \bar{J}_A^{(t)}) \right) - \begin{cases} \kappa'''' \bar{c}_A & \text{or} \\ \kappa'' \left( \bar{c}_A^2 + \bar{c}_A^{t^2} \right) & \end{cases} \]
Semi-empirical expressions from the turbulent mass flux

- **Eddy Diffusivity**

  The first law of diffusion (Fick's law)

  \[ \bar{\rho}'_{AB} = \mathcal{D}_{AB} \frac{d\bar{c}_A}{dy} \]

  As is the case with the eddy viscosity and the eddy thermal conductivity, the eddy diffusivity is not a physical property characteristic of the fluid, but depends on position, direction, and the nature of the flow field.

  The eddy diffusivity and the eddy kinematic viscosity \( \nu(t) = \frac{\nu(t)}{\rho} \) have the same dimensions.

  \[ Sc = \frac{\nu(t)}{\mathcal{D}(t)} \]
The Mixing-Length Expression of Prandtl and Taylor

According to the mixing-length theory of Prandtl, momentum, energy, and mass are all transported by the same mechanism.

We may write:

\[ j_{Ay}^{(n)} = -p \left( \frac{d \bar{v}_x}{dy} \right) \left( \frac{d \bar{c}_A}{dy} \right) \]
Enhancement of mass transfer by a first-order reaction in turbulent flow

We may study the effect of the reaction on the rate of mass transfer at the wall for steadily driven turbulent flow in a tube, where the wall (of material A) is slightly soluble in the fluid (a liquid B) flowing through the tube. Material A dissolves in liquid B and then disappears by a first-order reaction.

For tube flow with axial symmetry and with $c_A$ independent of time:

$$\bar{D}_z \frac{\partial \bar{c}_A}{\partial z} = \frac{1}{r} \frac{\partial}{\partial r} \left( r(D_{AB} + D_{AB}^{(n)}) \frac{\partial \bar{c}_A}{\partial r} \right) - k_i \bar{c}_A$$

Here we have made the customary assumption that the axial transport by both molecular and turbulent diffusion can be neglected.

\[
+ D_{AB} \frac{\partial \bar{c}_A}{\partial r} \bigg|_{r=R} = k_i (c_{A0} - \bar{c}_{A, axis})
\]

where $c_{A0}$ and $c_A$ are the concentrations at the tube axis.
The quantity \( k_c \) is a mass transfer coefficient, analogous to the heat transfer coefficient \( h \).

As a first approximation to be zero, assuming that the reaction is sufficiently rapid that the diffusing species never reaches the tube axis. After analyzing the system under this assumption, we will relax the assumption and give computations for a wider range of reaction rates.
The dimensionless reactant concentration

\[ C = \frac{c_A}{c_{A0}} \]

Then for large \( z \), the concentration will be independent of \( z \):

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r(D_{AB} + D_{AB}^{(n)}) \frac{\partial C}{\partial r} \right) = k''_1 C
\]

This equation may be multiplied by \( r \) and to give:

\[
k_c R - r D_{AB} + D_{AB}^{(n)} \frac{\partial C}{\partial r} = k''_1 \int_r^R \tilde{r} C(\tilde{r}) d\tilde{r}
\]

\( r = 0 \)

Then a second integration from \( r = 0 \) to \( r = R \) gives:

\[
k_c R \int_0^R \frac{1}{r(D_{AB} + D_{AB}^{(n)})} dr - 1 = k''_1 \int_0^R \frac{1}{r(D_{AB} + D_{AB}^{(n)})} \left[ \int_r^R \tilde{r} C(\tilde{r}) d\tilde{r} \right] d\tilde{r}
\]

boundary conditions: \( C = 0 \) at \( r = 0 \) and \( C = 1 \) at \( r = R \)
Next step is introducing the variable 
\[ y = R - r, \]
since the region of interest is right next to the wall.

\[
k_i R \int_0^R \frac{1}{(R - y)(D_{AB} + D_{AB}^{(v)})} \, dy - 1 = k''_i \int_0^R \frac{1}{(R - y)(D_{AB} + D_{AB}^{(v)})} \left[ \int_y^y (R - \bar{y}) C(\bar{y}) \, d\bar{y} \right] \, dy
\]

We can use the fact that the turbulent diffusivity in the neighborhood of the wall is proportional to the third power of the distance from the wall.

When the integrals are rewritten in terms of \( \ell = y/ \ell \), we get the dimensionless equation:

\[
\frac{1}{2} \left( \frac{k_i D}{D_{AB}} \right) \left( \frac{D_{AB}}{\nu} \right) \int_0^1 \frac{1}{(D_{AB}/\nu) + K \sigma^3} \, d\sigma - 1 = \left( \frac{k''_i R^2}{\nu} \right) \int_0^1 \frac{1}{(D_{AB}/\nu) + K \sigma^3} \left[ \int_0^\sigma C(\sigma) \, d\sigma \right] \, d\sigma
\]
The equation

\[ C = \exp(-Sh\sigma/2) \]

substitution of this solution into

\[
\frac{1}{2} \left( \frac{k_c D}{D_{AB}} \right) \int_0^1 \frac{1}{(D_{AB}/\nu) + K\sigma^3} d\sigma - 1 = \left( \frac{k'' R^2}{\nu} \right) \int_0^1 \frac{1}{(D_{AB}/\nu) + K\sigma^3} \left[ \int_0^\sigma C(\sigma) d\sigma \right] d\sigma
\]

gives after straightforward integration

\[
\frac{1}{2} \frac{Sh}{Sc} I_0 - 1 = 2 \frac{Rx}{Sh} I_0 - 2 \frac{Rx}{Sh} I_1
\]
in which

\[
I_0 = \int_0^1 \frac{1}{Sc^{-1} + K\sigma^3} d\sigma
\]

\[
I_1 = \int_0^1 \exp(-Sh\sigma/2) \frac{1}{Sc^{-1} + K\sigma^3} d\sigma
\]
For a better analysis of the enhancement problem, we use

$$
\bar{v}_z \frac{\partial \bar{c}_A}{\partial Z} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \left( \phi_{AB} + \phi_{AB}^{(1)} \right) \frac{\partial \bar{c}_A}{\partial r} \right) - k_1^m \bar{c}_A
$$

to get a more complete differential equation for C:

$$
\bar{v}_z \frac{\partial C}{\partial Z} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \left( \phi_{AB} + \phi_{AB}^{(1)} \right) \frac{\partial C}{\partial r} \right) - k_1'' C
$$