Transport Phenomena

Interphase Transport in Isothermal Systems
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1. Definition of friction factors
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4. Friction factors for packed columns
1. Definition of Friction Factors

- We consider the steadily driven flow of a fluid of constant density in one of two systems: (a) the fluid flows in a straight conduit of uniform cross section; (b) the fluid flows around a submerged object that has an axis of symmetry parallel to the direction of the approaching fluid. There will be a force $F_{fs}$ exerted by the fluid on the solid surfaces. It is convenient to split this force into two parts:
  - $F_s$ - force that would be exerted by the fluid event if it were stationary
  - $F_k$ - the additional force associated with the motion of the fluid

In systems of type (a), $F_k$ points in the same direction as the average velocity $<v>$ in the conduit, and in systems of type (b), $F_k$ points in the same direction as the approach velocity $v_\infty$. For both types of systems we state that the magnitude of the force $F_A$ is proportional to a characteristic area $A$ and a characteristic kinetic energy $K$ per unit volume:

$$F_k = AKf$$

where $f$ is called the friction factor.

This equation is only definition of $f$. This useful because the dimensionless quantity $f$ can be given as a relatively simple function of the Reynolds number and the system shape.
1. Definition of Friction Factors

Clearly, for any given flow system, $f$ is not defined until $A$ and $K$ are specified. (a) For flow in conduits, $A$ is usually taken to be the wetted surface, and $K$ is taken to be $\frac{1}{2} \rho (v)^2$. Specifically, for circular tubes of radius $R$ and length $L$ :

$$F_k = (2\pi RL)\left(\frac{1}{2} \rho (v)^2\right)f$$

Generally, the quantity measured is not $F_k$, but rather the pressure difference $p_0 - p_L$ and the elevation difference $h_0 - h_L$. A force balance on the fluid between 0 and $L$ in the direction of flow gives for fully developed flow

$$F_k = [(p_0 - p_L) + \rho g (h_0 - h_L)]\pi R^2 = (\mathcal{P}_0 - \mathcal{P}_L) \pi R^2$$

Elimination of $F_k$ between the last two equations then gives

$$f = \frac{1}{4} \left(\frac{D}{L}\right)\left(\frac{\mathcal{P}_0 - \mathcal{P}_L}{\frac{1}{2} \rho (v)^2}\right)$$

in which $D = 2R$ is the tube diameter. The quantity $f$ is sometimes called the Fanning friction factor
1. Definition of Friction Factors

(b) For flow around submerged objects, the characteristic area $A$ is usually taken to be the area obtained by projecting the solid onto a plane perpendicular to the velocity of the approaching fluid; the quantity $K$ is taken to be $\frac{1}{2} \rho v^2_{\infty}$, where $v_{\infty}$ is the approach velocity of the fluid at a large distance from the object. For example, for flow around a sphere of radius $R$, we define $f$ by the equation

$$F_k = (\pi R^2)(\frac{1}{2} \rho v^2_{\infty})f$$

If it is not possible to measure $F_k$, then we can measure the terminal velocity of the sphere when it falls through the fluid (in that case, $v_{\infty}$ has to be interpreted as the terminal velocity of the sphere). For the steady-state fall of a sphere in a fluid, the force $F_k$ is just counterbalanced by the gravitational force on the sphere less the buoyant force:

$$F_k = \frac{4}{3} \pi R^3 \rho_{sp} g - \frac{4}{3} \pi R^3 \rho g$$

Elimination of $F_k$ between the last two equations then gives

$$f = \frac{4}{3} \frac{gD}{v^2_{\infty}} \left( \frac{\rho_{sp} - \rho}{\rho} \right)$$

This expression can be used to obtain $f$ from terminal velocity data.
2. Friction Factors For Flow In Tubes

In turbulent flow the force may be a function of time, not only because of the turbulent fluctuations, but also because of occasional ripping off of the boundary layer from the wall, which results in some distances with long time scales. In laminar flow it is understood that the force will be independent of time.

\[ F_k(t) = \int_0^L \int_0^{2\pi} \left( -\mu \frac{\partial v_z}{\partial r} \right) \Bigg|_{r=R} R \, d\theta \, dz \]

we can introduce the dimensionless quantities from chapter 3.7:

\[ \ddot{r} = r/D, \ddot{z} = z/D, \dot{v}_z = v_z/(\langle v_z \rangle), \dot{t} = \langle v_z \rangle t/D, \ddot{P} = (P_0 - P)/\rho(\langle v_z \rangle)^2, Re = D(\langle v_z \rangle \rho/\mu). \]

so we have:

\[ f(\ddot{t}) = \frac{1}{\pi} \frac{D}{L} \frac{1}{Re} \int_0^{L/D} \int_0^{2\pi} \left( -\frac{\partial \ddot{v_z}}{\partial \ddot{r}} \right) \Bigg|_{\ddot{r}=1/2} d\theta \, d\ddot{z} \]
2. Friction Factors For Flow In Tubes

\[ f(t) = \frac{1}{\pi} \frac{D}{L} \frac{1}{\text{Re}} \int_0^{L/D} \int_0^{2\pi} \left( -\frac{\partial \tilde{v}_z}{\partial \tilde{r}} \right) \bigg|_{\tilde{r} = 1/2} d\theta \, d\tilde{z} \]

This relation is valid for laminar or turbulent flow in circular tubes. We see that for flow systems in which the drag depends on viscous forces alone ("form drag") the product of \( f/\text{Re} \) is essentially a dimensionless velocity gradient averaged over the surface.

Boundary conditions:

\[
\begin{align*}
\tilde{r} = \frac{1}{2}, & \quad \tilde{v} = 0 \quad \text{for } z > 0 \\
\tilde{z} = 0, & \quad \tilde{v} = \delta_z \\
\tilde{r} = 0 \text{ and } \tilde{z} = 0, & \quad \tilde{\Phi} = 0 \\
\tilde{v} = \tilde{v}(\tilde{r}, \theta, \tilde{z}, \tilde{t}; \text{Re}) \\
\tilde{\Phi} = \tilde{\Phi}(\tilde{r}, \theta, \tilde{z}, \tilde{t}; \text{Re})
\end{align*}
\]

That is, the functional dependence of \( \mathbf{v} \) and \( P \) must, in general, include all the dimensionless variables and the one dimensionless group appearing in the differential equations. No additional dimensionless groups enter via the preceding boundary conditions. As a consequence, \( \partial \tilde{v}_z/\partial \tilde{r} \) must likewise depend on \( r, \theta, z, t, \) and \( \text{Re} \). When \( \partial \tilde{v}_z/\partial \tilde{r} \, d\tilde{r} \) is evaluated at \( r = 1/2 \) and then integrated over \( z \) and \( \theta \) in equation of the top, the result depends only on \( t, \text{Re}, \) and \( L/D \) (the latter appearing in the upper limit in the integration over \( z \)). Therefore we are led to the conclusion that \( f(t) = f(\text{Re}, L/D, t) \), which, when time averaged, becomes

\[ f = f(\text{Re}, L/D) \]

when the time average is performed over an interval long enough to include any long-time turbulent disturbances. The measured friction factor then depends only on the Reynolds number and the length-to-diameter ratio.
2. Friction Factors For Flow In Tubes

The *laminar* curve on the friction factor chart is merely a plot of the *Hagen-Poiseuille* equation.

\[
f = \frac{16}{Re} \begin{cases} 
Re < 2100 & \text{stable} \\
Re > 2100 & \text{usually unstable} 
\end{cases}
\]

in which \( Re = \frac{D \langle \bar{v} \rangle \rho}{\mu} \),

where:
- \( D \) is the pipe diameter,
- \( \langle \bar{v} \rangle \) is the mean velocity,
- \( \rho \) is the fluid density,
- \( \mu \) is the fluid viscosity.

The friction factor graph shows the relationship between the Reynolds number and the friction factor for both laminar and turbulent flow conditions.
2. Friction Factors For Flow In Tubes

Turbulent curves have been constructed by using experimental data. Some analytical curve-fit expressions are also available. For turbulent flow in noncircular tubes it is common to use the following empiricism: First we define a "mean hydraulic radius" $R_h$ as follows:

$$R_h = \frac{S}{Z}$$

In which $S$ is the cross section of the conduit and $Z$ is the wetted perimeter.

$$f = \frac{1}{4} \left( \frac{D}{L} \right) \left( \frac{P_0 - P_L}{\frac{1}{2} \rho \langle v \rangle^2} \right) + f(t) = \frac{\int_0^L \int_0^{2\pi} \left( -\mu \frac{\partial v_z}{\partial r} \right) \bigg|_{r=R} \, R \, d\theta \, dz}{(2\pi RL) \left( \frac{1}{2} \rho \langle v_z \rangle^2 \right)}$$

we get:

$$f = \left( \frac{R_h}{L} \right) \left( \frac{P_0 - P_L}{\frac{1}{2} \rho \langle v_z \rangle^2} \right)$$

$$Re_h = \frac{4R_h \langle v_z \rangle \rho}{\mu}$$

This estimation method is not for laminar flow!!!
2. Friction Factors For Flow In Tubes

Example

What pressure gradient is required to cause diethylaniline, C_6H_5N(C_2H_5)_2, to flow in a horizontal, smooth, circular tube of inside diameter D = 3 cm at a mass rate of 1028 g/s at 20°C? At this temperature the density of diethylaniline is 0.935 g/cm^3 and its viscosity is \( \mu = 1.95 \)

Solution

The Reynolds number for the flow is

\[
Re = \frac{D(v_z)\rho}{\mu} = \frac{ Dw }{(\pi D^2/4)\mu} = \frac{4w}{\pi D \mu}
\]

\[
= \frac{4(1028 \text{ g/s})}{\pi(3 \text{ cm})(1.95 \times 10^{-2} \text{ g/cm \cdot s})} = 2.24 \times 10^4
\]

We find from the table that for this Reynolds number the friction factor \( f \) has a value of 0.0063 for smooth tubes. Hence the pressure gradient required to maintain the flow is

\[
\frac{p_0 - p_L}{L} = \left( \frac{4}{D} \right) \left( \frac{1}{2} \rho(v_z)^2 \right) f = \frac{2}{D} \rho \left( \frac{4w}{\pi D^2 \rho} \right)^2 f
\]

\[
= \frac{32w^2f}{\pi^2 D^5 \rho} = \frac{(32)(1028)^2(0.0063))}{\pi^2(3.0)^5(0.935)}
\]

\[
= 95(\text{dyne/cm}^2)/\text{cm} = 0.071(\text{mm Hg})/\text{cm}
\]
3. Friction Factors For Flow Around Spheres

Recall from chapter 2.6 that the total force acting in the \( z \) direction on the sphere can be written as the sum of a contribution from the normal stresses \((F_n)\) and one from the tangential stresses \((F_t)\). One part of the normal-stress contribution is the force that would be present even if the fluid were stationary, \( F_s \). Thus the "kinetic force," associated with the fluid motion, is

\[
F_k = (F_n - F_s) + F_t = F_{\text{form}} + F_{\text{friction}}
\]

The forces associated with the form drag and the friction drag are then obtained from

\[
F_{\text{form}}(t) = \int_0^{2\pi} \int_0^{\pi} (-\mathcal{P}|_{r=R} \cos \theta) R^2 \sin \theta \, d\theta \, d\phi
\]

\[
F_{\text{friction}}(t) = \int_0^{2\pi} \int_0^{\pi} \left(-\mu \left[ r \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right] \right) |_{r=R} \sin \theta \, R^2 \sin \theta \, d\theta \, d\phi
\]

Since \( v_r \) is zero everywhere on the sphere surface, the term containing \( dv_r/d\theta \) is zero. If now we split \( f \) into two parts as follows

\[
f = f_{\text{form}} + f_{\text{friction}}
\]

then, from the definition we get:

\[
f_{\text{form}}(t) = \frac{2}{\pi} \int_0^{2\pi} \int_0^{\pi} (-\mathcal{P}|_{r=1} \cos \theta) \sin \theta \, d\theta \, d\phi
\]

\[
f_{\text{friction}}(t) = -\frac{4}{\pi} \frac{1}{\text{Re}} \int_0^{2\pi} \int_0^{\pi} \left[ \frac{\partial}{\partial \tilde{r}} \left( \tilde{v}_\theta \right) \right] |_{\tilde{r}=1} \sin^2 \theta \, d\theta \, d\phi
\]
3. Friction Factors For Flow Around Spheres

The friction factor and Reynolds number is expressed here in terms of dimensionless variables:

\[ \tilde{f} = \frac{f}{\rho v_\infty^2}, \quad \tilde{v}_\theta = \frac{v_\theta}{v_\infty}, \quad \tilde{r} = \frac{r}{R}, \quad \tilde{v} = \frac{v_\infty t}{R}, \quad \text{Re} = \frac{Dv_\infty \rho}{\mu} = \frac{2Rv_\infty \rho}{\mu} \]

We are defining the boundary conditions:

\[ \tilde{r} = 1, \quad \tilde{v}_r = 0 \quad \text{and} \quad \tilde{v}_\theta = 0 \]
\[ \tilde{r} = \infty, \quad \tilde{v}_z = 1 \]
\[ \tilde{r} = \infty, \quad \tilde{P} = 0 \]

Because no additional dimensionless groups enter via the boundary and initial conditions, we know that the dimensionless pressure and velocity profiles will have the following form:

\[ \tilde{P} = \tilde{P}(\tilde{r}, \theta, \phi, \tilde{t}; \text{Re}) \quad \tilde{v} = \tilde{v}(\tilde{r}, \theta, \phi, \tilde{t}; \text{Re}) \]

Tube flow and flow around a sphere behave quite differently. Several points of difference between the two systems are:

**Flow in Tubes**
- Rather well defined laminar-turbulent transition at about Re = 2100
- The only contribution to \( f \) is the friction drag
- No boundary layer separation

**Flow Around Spheres**
- No well defined laminar-turbulent transition
- Contributions to \( f \) from both friction and form drag
- There is a kink in the \( f \) vs. Re curve associated with a shift in the separation zone
For the *creeping flow region*, the drag force is given by *Stokes' law*, which is a consequence of solving the continuity equation and the Navier-Stokes equation of motion without the $\rho Dv/Dt$ term. Stokes' law can be rearranged:

$$F_k = (\pi R^2)(\frac{1}{2}\rho v_\infty^2)\left(\frac{24}{Dv_\infty \rho / \mu}\right)$$

Hence for *creeping flow* around a sphere

$$f = \frac{24}{Re} \quad \text{for } Re < 0.1$$

and this is the straight-line asymptote as $Re \rightarrow 0$ of the friction factor curve in picture.

Friction factor (or drag coefficient) for spheres moving relative to a fluid with a velocity $v_\infty$. 

![Friction factor curve](image)
3. Friction Factors For Flow Around Spheres

Example

Glass spheres of density $\rho_{\text{sph}} = 2.62 \text{ g/cm}^3$ are to be allowed to fall through liquid $\text{CC}_4$ at $20^\circ \text{C}$ in an experiment for studying human reaction times in making time observations with stopwatches and more elaborate devices. At this temperature the relevant properties of $\text{CC}_4$ are $\rho = 1.59 \text{ g/cm}^3$ and $\mu = 9.58 \text{ millipoises}$. What diameter should the spheres be to have a terminal velocity of about $65 \text{ cm/s}$?

Solution

To find the sphere diameter, we have to solve previous equation for $D$. However, in this equation one has to know $D$ in order to get $f$; and $f$ is given by the solid curve in picture. A trial-and-error procedure can be used, taking $f = 0.44$ as a first guess.

But another way is by solving previous equation for $f$ and then note that $f/\text{Re}$ is a quantity independent of $D$ we have:

$$f = \frac{4gD}{3\nu^2}\left(\frac{\rho_{\text{sph}} - \rho}{\rho}\right)$$

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But another way is by solving previous equation for $f$ and then note that $f/\text{Re}$ is a quantity independent of $D$ we have:

$$\frac{f}{\text{Re}} = \frac{4g\mu}{3\nu^3}\left(\frac{\rho_{\text{sph}} - \rho}{\rho}\right)$$

The quantity on the right side can be calculated with the information above, and we call it $C$. Hence we have two simultaneous equations to solve:

$$a) \quad f = C \text{ Re}$$

$$b) \quad f = f(\text{Re})$$

First equation is a straight line with slope of unity on the log-log plot of $f$ versus $\text{Re}$. For the problem at hand we have:

$$C = \frac{4}{3} \frac{(980)(9.58 \times 10^{-3})}{(1.59)(65)^3} \left(\frac{2.62 - 1.59}{1.59}\right) = 1.86 \times 10^{-5}$$

Hence at $\text{Re} = 10^5$, according to Eq. $a$, $f = 1.86$. The line of slope 1 passing through $f = 1.86$ at $\text{Re} = 10^5$ is shown in Picture. This line intersects the curve of Eq. $b$ (the curve of picture.) at $\text{Re} = Dv_{\infty}/\mu = 2.4 \times 10^4$. The sphere diameter is then found to be

$$D = \frac{\text{Re} \mu}{\rho v_{\infty}} = \frac{(2.4 \times 10^4)(9.58 \times 10^{-3})}{(1.59)(65)} = 2.2 \text{ cm}$$