TRANSPORT PHENOMENA
MOMENTUM TRANSPORT

Macroscopic Balances for Isothermal Flow Systems
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1. The macroscopic mass balance
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Macroscopic Mass Balance

We consider a macroscopic flow system with fluid entering at plane 1 with cross section $S_1$ and leaving at plane 2 with cross section $S_2$.

We introduce two assumptions:

1. at the planes 1 and 2 the velocity is perpendicular to the relevant cross section,

2. at planes 1 and 2 the density and other physical properties are uniform over the cross section.
Macroscopic Mass Balance

The law of conservation of mass for this system:

\[
\frac{d}{dt} m_{\text{tot}} = \rho_1(v_1)S_1 - \rho_2(v_2)S_2
\]

where:

- \(v_1\) is an average velocity at the entry plane 1,
- \(v_2\) is an average velocity at the exit plane 2,
- \(S_1, S_2\) are cross sections,
- \(m_{\text{tot}}\) is a total mass of fluid contained in the system between planes 1 and 2: \(m_{\text{tot}} = \int \rho dV\)
Macroscopic Mass Balance

We introduce the symbol \( w \) for the mass rate of flow:

\[
  w = \rho(v)S
\]

and the notation of \( \Delta w \):

\[
  \Delta w = w_2 - w_1 \text{ (exit value minus entrance value)}
\]

Then the unsteady-steady macroscopic mass balance becomes:

\[
  \frac{d}{dt} m_{tot} = -\Delta w
\]

When the total mass of fluid does not change with time, then we get the steady-state macroscopic mass balance:

\[
  \Delta w = 0
\]
Macroscopic Momentum Balance

We apply the law of conservation of momentum to the same system, using two additional assumptions:

We introduce four assumptions:

1. at the planes 1 and 2 the velocity is perpendicular to the relevant cross section,
2. at planes 1 and 2 the density and other physical properties are uniform over the cross section,
3. the forces associated with the stress tensor $\tau$ are neglected at planes 1 and 2,
4. the pressure does not vary over the cross section at planes 1 and 2.
Macroscopic Momentum Balance

The law of conservation of mass for this system:

\[
\frac{d}{dt} P_{\text{tot}} = \rho_1 \langle v_1^2 \rangle S_1 u_1 - \rho_2 \langle v_2^2 \rangle S_2 u_2 + p_1 S_1 u_1 - p_2 S_2 u_2 + F_{\text{g-\text{f}}} + m_{\text{tot}} g
\]

rate of increase of momentum in at plane 1
rate of momentum out at plane 2
pressure force on fluid at plane 1
pressure force on fluid at plane 2
force of solid surface on fluid
force of gravity on fluid

Since momentum is a vector quantity, each term must be a vector. We use unit vectors \( u_1 \) and \( u_2 \) to present the direction of flow at planes.

where:

- \( P_{\text{tot}} \) is the total momentum in the system, \( P_{\text{tot}} = \int \rho v dV \)
- the subscript "s \( \rightarrow \) f" serves as a reminder of the direction of the force.
Macroscopic Momentum Balance

By introducing the symbols for the mass rate of flow and the $\Delta$ symbol we finally get for the unsteady-state macroscopic momentum balance:

$$\frac{d}{dt} P_{\text{tot}} = -\Delta \left( \langle v^2 \rangle w + pS \right) u + F_{s\rightarrow f} + m_{\text{tot}} g$$

If the total amount of momentum in the system does not change with time, then we get the steady-state macroscopic momentum balance:

$$F_{f\rightarrow s} = -\Delta \left( \langle v^2 \rangle w + pS \right) u + m_{\text{tot}} g$$
Macroscopic Angular Momentum Balance

The development of the macroscopic angular momentum balance parallels that for the (linear) momentum balance in the previous section. All we have to do is to replace "momentum" by "angular momentum" and "force" by "torque".

To describe the angular momentum and torque we have to select an origin of coordinates "O", and the locations of the midpoints at plane 1 and 2 with respect to this origin are given by the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. 
Macroscopic Angular Momentum Balance

We apply the momentum balance using the same 4 assumptions introduced in previous sections.

The unsteady-state macroscopic angular momentum balance may now be written as:

\[
\frac{d}{dt} \mathbf{L}_{\text{tot}} = \rho_1 \langle v_1^2 \rangle S_1 [r \times u_1] - \rho_2 \langle v_2^2 \rangle S_2 [r \times u_2] + p_1 S_1 [r \times u_1] - p_2 S_2 [r \times u_2] + \mathbf{T}_{\text{s-off}} + \mathbf{T}_{\text{ext}}
\]

where:

– \( \mathbf{L}_{\text{tot}} = \int \rho [r \times \mathbf{v}] dV \) is the total angular momentum,

– \( \mathbf{T}_{\text{ext}} = \int [r \times \rho \mathbf{g}] dV \) is the torque on the fluid in the system from the gravitational force.
Macroscopic Angular Momentum Balance

Macroscopic angular momentum balance can also be written as:

$$\frac{d}{dt} L_{tot} = -\Delta \left( \frac{\langle v^2 \rangle}{\langle v \rangle} w + pS \right) [r \times u] + T_{s\rightarrow f} + T_{ext}$$

Finally, the **steady-state macroscopic angular momentum balance** is:

$$T_{f\rightarrow s} = -\Delta \left( \frac{\langle v^2 \rangle}{\langle v \rangle} w + pS \right) [r \times u] + T_{ext}$$

This gives the torque exerted by the fluid on the solid surfaces.
Macroscopic Mechanical Energy Balance

To get the equation balance, we must integrate the equation of change of mechanical energy over the volume of the flow system and use the same assumptions (1-4) used above:

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho v^2 + \rho \Phi \right) = - \left( \nabla \cdot \left( \frac{1}{2} \rho v^2 + \rho \Phi \right) \right) v - (\nabla \cdot p v) - p(-\nabla \cdot v) - (\nabla \cdot [\tau \cdot v]) - (-\tau : \nabla v)$$

The result is the unsteady-state macroscopic mechanical energy balance (sometimes called the engineering Bernoulli equation):
Macroscopic Mechanical Energy Balance

Where:

\[ K_{\text{tot}} = \int \frac{1}{2} \rho v^2 \, dV \quad \text{and} \quad \Phi_{\text{tot}} = \int \rho \Phi \, dV \] are the total kinetic and potential energies within the system.

Since, at the system entrance (plane 1), the force \( \rho_1 S_1 \) multiplied by the velocity \( \langle v_1 \rangle \) gives the rate at which the surroundings do work on the fluid, we introduce the symbol \( W_m \), which is the work done by the surroundings on the fluid by means of moving surfaces.

The macroscopic mechanical energy balance may now be written more compactly:

\[
\frac{d}{dt} (K_{\text{tot}} + \Phi_{\text{tot}}) = -\Delta \left( \frac{1}{2} \langle v^3 \rangle + \hat{\Phi} + \frac{p}{\rho} \right) w + W_m - E_c - E_v
\]
Macroscopic Mechanical Energy Balance

If the total kinetic plus potential energy in the system is not changing with time, we get:

$$\Delta \left( \frac{1}{2} \frac{\langle v^3 \rangle}{\langle v \rangle} + gh + \frac{p}{\rho} \right) w = W_m - E_c - E_v$$

Which is the steady-state macroscopic mechanical energy balance.
Estimation of the Viscous Loss

In the macroscopic mechanical energy balance appears the term $E_v$, which is the viscous dissipation term. The general expression for $E_v$ is

$$E_v = -\int_{V(t)} (\tau : \nabla \mathbf{v}) \, dV$$

For incompressible Newtonian fluids may be used to rewrite $E_v$ as:

$$E_v = \int \mu \Phi_v dV$$

Which shows that it is the integral of the local rate of viscous dissipation over the volume of the entire flow system.
Estimation of the Viscous Loss

In steady-state flow we prefer to work with the quantity \( \hat{E}_v = E_v/w \), in which \( w = \rho \langle v \rangle S \) is the mass rate of flow passing through any cross section of the flow system. If we select the reference velocity \( v_0 \) to be \( \langle v \rangle \) and the reference length \( l_0 \) to be \( \sqrt{S} \), then:

\[
\hat{E}_v = \frac{1}{2} \langle v \rangle^2 e_v
\]

In which \( e_v \), the friction loss factor, is a function of a Reynolds number and relevant dimensionless geometrical ratios. The factor \( \frac{1}{2} \) has been introduced in keeping with the form of several related equations.
Estimation of the Viscous Loss

EXAMPLE 7.5-1

What is the required power output from the pump at steady state in the system? The water ($\rho = 62.4 \text{ lb}_m/\text{ft}^3; \mu = 1.0 \text{ cp}$) is to be delivered to the upper tank at a rate of 12 $\text{ft}^3/\text{min}$. All of the piping is 4 in. internal diameter smooth circular pipe.
Estimation of the Viscous Loss

**EXAMPLE 7.5-1**

The average velocity in the pipe is

\[ w = \rho \langle v \rangle S \Rightarrow \langle v \rangle = \frac{w/\rho}{\pi R^2} = \frac{12/60}{\pi (1/6)^2} = 2.30 \text{ ft/s} \]

and the Reynolds number is

\[ \text{Re} = \frac{D \langle v \rangle \rho}{\mu} = \frac{(1/3) \cdot 2.30 \cdot 62.4}{1.0 \cdot (6.72 \cdot 10^{-4})} = 7.11 \cdot 10^4 \]

Hence the flow is turbulent.
Estimation of the Viscous Loss

EXAMPLE 7.5-1

The contribution to $\hat{E}_v$ from the various lengths of pipe will be

$$
\sum_i \left( \frac{1}{2}v^2 \frac{L}{R_h} f \right)_i = \frac{2v^2f}{D} \sum_i L_i =
$$

$$
= \frac{2(2.30)^2(0.0049)}{(1/3)} (5 + 300 + 100 + 120 + 20) = (0.156)(545) = 85 \text{ ft}^2/\text{s}^2
$$

The contribution to $\hat{E}_v$ from the sudden contraction, the three 90° elbows, and the sudden expansion will be

$$
\sum_i \left( \frac{1}{2}v^2e_i \right)_i = \frac{1}{2}(2.30)^2(0.45 + 3(1) + 1) = 8 \text{ ft}^2/\text{s}^2
$$
**Estimation of the Viscous Loss**

**EXAMPLE 7.5-1**

Then from the steady-state macroscopic mechanical energy balance in the approximate form used for turbulent flow:

$$\frac{1}{2}(v_2^2 - v_1^2) + g(z_2 - z_1) + \int_{p_1}^{p_2} \frac{1}{\rho} dp = \hat{W}_m - \sum_i \left( \frac{1}{2}v_i^2 \frac{L}{R_h} f \right)_i - \sum_i \left( \frac{1}{2}v_i^2 e_v \right)_i$$

we get

$$0 + (32.2)(105 - 20) + 0 = \hat{W}_m - 85 - 8$$

Solving for \(\hat{W}_m\) we get

$$\hat{W}_m = 2740 + 85 - 8 \approx 2830 \text{ ft}^2/\text{s}^2$$

This is the work (per unit mass of fluid) done on the fluid in the pump. The mass rate of flow is

$$\dot{w} = (12/60)(62.4) = 12.5 \text{ lb}_m/\text{s}$$

Consequently

$$\hat{W}_m = \dot{w} \hat{W}_m = (12.5)(88) = 1100 \text{ ft lb}_f/\text{s} = 2 \text{ hp} = 1.5 \text{ kW}$$

Which is the power delivered by the pump.